Formal introduction to nonstandard analysis

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1. Nonstandard analysis

There are several different formulations of nonstandard analysis. This thesis adopts the set-theoretical approach based on superstructures instituted by Robinson and Zakon [3] and follows the up-to-date description by Chang and Keisler [1].

For any set X, let $S(X)$ denote the set of all subsets of X. The *superstructure* over X, denoted by $V(X)$, is defined by the following recursion:

$$V_0(X) = X, \quad V_{n+1}(X) = V_n(X) \cup S(V_n(X)), \quad V(X) = \bigcup_{n \in \mathbf{N}} V_n(X),$$

where $\mathbf{N}$ is the set of natural numbers. The set X is called a *base set* if $\emptyset \notin X$ and for all $x \in X$ we have $x \cap V(X) = \emptyset$.

The language $\mathcal{L}$ which describes $V(X)$ consists of logical connectives $\neg$, $\wedge$, $\vee$, $\Rightarrow$, quantifiers $\forall$, $\exists$, individual variables x',x'', $\ldots$, individual constants C_u for all $u \in V(X)$, and two binary predicate constants $=$, $\in$. A *formula* of $\mathcal{L}$ is constructed from the above constituents in the usual way. We will use the following abbreviations, called *bounded quantifiers*: $(\forall x \in y)\phi$ means $(\forall x)[x \in y \Rightarrow \phi]$, $(\exists x \in y)\phi$ means $(\exists x)[x \in y \wedge \phi]$. A *bounded formula* is a formula in which every quantifier occurs as a bounded quantifier. We will write $\phi[u_1, ..., u_n]$ for $\phi(C_{u_1}, ..., C_{u_n})$.

For any formula ϕ in $\mathcal{L}$, the relation $V(X) \models \phi$ is defined by the following rules:

(i) $V(X) \models C_u = C_v$ if and only if u and v are identical.

(ii) $V(X) \models C_u \in C_v$ if and only if u is an element of v.

(iii) $V(X) \models \neg\phi$ if and only if $V(X) \models \phi$ does not hold.

(iv) $V(X) \models \phi_1 \wedge \phi_2$ if and only if $V(X) \models \phi_1$ and $V(X) \models \phi_2$.

(v) $V(X) \models \phi_1 \vee \phi_2$ if and only if $V(X) \models \phi_1$ or $V(X) \models \phi_2$.

(vi) $V(X) \models \phi_1 \Rightarrow \phi_2$ if and only if $V(X) \models \phi_1$ then $V(X) \models \phi_2$.

(vii) $V(X) \models (\forall x)\phi(x)$ if and only if $V(X) \models \phi[u]$ for all u in $V(X)$.

(viii) $V(X) \models (\exists x)\phi(x)$ if and only if $V(X) \models \phi[u]$ for some u in $V(X)$.

A *nonstandard universe* is a triple $V(X), V(Y), \star$ consisting of superstructures $V(X)$, $V(Y)$, and a map $\star : V(X) \to V(Y)$ satisfying the following conditions (i)–(iii):

(i) X and Y are infinite base sets.

(ii) **(Transfer Principle)** The map $\star : a \mapsto {}^\star a$ is an injective mapping from $V(X)$ into $V(Y)$, and for any bounded formula $\phi(x_1, \ldots, x_n)$ in $\mathcal{L}$,

$$V(X) \models \phi[u_1, \ldots, u_n] \quad \text{if and only if} \quad V(Y) \models \phi[{}^\star u_1, \ldots, {}^\star u_n]$$

for any $u_1, \ldots, u_n$ in $V(X)$.

(iii) ${}^\star X = Y$.

An element $u \in V(Y) \setminus Y$ is called an *internal set* if there is $x \in V(X)$ such that $u \in {}^\star x$. Let α be a cardinal. A nonstandard universe $V(X), V(Y), \star$ is said to be *α-saturated* if it satisfies the following condition:

(iv) **(Saturation Principle)** Every family of less than α internal sets with the finite intersection property has nonempty intersection.

In this thesis, we always work with a nonstandard universe $V(X), V(Y), \star$ which is α-saturated with $\mathrm{card}(V(X)) < \alpha$; such a nonstandard universe is said to be *polysaturated.* We also assume that the base X includes the complex numbers $\mathbf{C}$ and any other structures under consideration such as given groups and Hilbert spaces.

For a set S, let ${}^\sigma S = \{ {}^\star s |\ s \in S\}$. We identify ${}^\star z$ with z for all $z \in \mathbf{C}$. Hence, ${}^\sigma S = S$ if S is a subset of $\mathbf{C}$, e.g., ${}^\sigma\mathbf{C} = \mathbf{C}$, ${}^\sigma\mathbf{R} = \mathbf{R}$ (the real numbers), ${}^\sigma\mathbf{Z} = \mathbf{Z}$ (the integers), and ${}^\sigma\mathbf{N} = \mathbf{N}$. Let $\mathbf{R}^+$, ${}^\star\mathbf{R}_0$, ${}^\star\mathbf{R}_0^+$, ${}^\star\mathbf{R}_\infty^+$, and ${}^\star\mathbf{N}_\infty$ denote the sets of positive real numbers, infinitesimal hyperreal numbers, positive infinitesimal hyperreal numbers, positive infinite hyperreal numbers and infinite hypernatural numbers, respectively. It is shown that ${}^\star\mathbf{N}_\infty = {}^\star\mathbf{N} \setminus \mathbf{N}$. We write $x \sim \infty$ if $x \in \mathbf{R}_\infty^+$, and $0 < x < \infty$ if $x \in \mathrm{fin}\,{}^\star\mathbf{R}^+ = {}^\star\mathbf{R}^+ \setminus {}^\star\mathbf{R}_\infty^+$. If $r \in {}^\star\mathbf{R}$ and $|r| < \infty$, the standard part of r is denoted by ${}^\circ r$. If $r \sim \infty$, we write ${}^\circ r = \infty$. Let $x, y \in {}^\star\mathbf{R}^+$. we say that x is of the *order* of y, in symbols $x \asymp y$, iff $0 < x/y < \infty$ and $0 < y/x < \infty$. We write $x \ll y$ if $x/y \approx 0$. For a hyperfinite ($\star$-finite) set F, let $|F|$ denote the internal cardinal number of F.

Let $(X, \mathcal{O})$ be a topological space. Let $\mathcal{O}_x$ denote the system of open neighborhoods of $x \in X$. The *monad* of $x \in X$ is the subset of ${}^\star X$ defined by $\mathrm{mon}_{\mathcal{O}}(x) = \bigcap\{ {}^\star O |\ O \in \mathcal{O}_x\}$. The set of *near standard* points is the subset of ${}^\star X$ defined by $\mathrm{ns}({}^\star X) = \bigcup\{\mathrm{mon}_{\mathcal{O}}(x)|\ x \in X\}$. It is shown that $(X, \mathcal{O})$ is Hausdorff if and only if $x \neq y$ implies $\mathrm{mon}_{\mathcal{O}}(x) \cap \mathrm{mon}_{\mathcal{O}}(y) = \emptyset$. Thus for any Hausdorff space $(X, \mathcal{O})$, we can define the equivalence relation $\approx_{\mathcal{O}}$ on $\mathrm{ns}\,{}^\star X$ so that $a {\approx_{\mathcal{O}}} b$ iff $a \in \mathrm{mon}_{\mathcal{O}}(x)$ and $b \in \mathrm{mon}_{\mathcal{O}}(x)$ for some $x \in X$.

Let $(V, \|\cdot\|)$ be an internal normed linear space. Define the subspaces $\mu(V, \|\cdot\|)$ and $\operatorname{fin}(V, \|\cdot\|)$ of V by

$$\mu(V, \|\cdot\|) = \{\xi \in V |\ \|\xi\| \approx 0\}, \qquad \operatorname{fin}(V, \|\cdot\|) = \{\xi \in V |\ \|\xi\| < \infty\}. \quad (1)$$

We often abbreviate them as $\mu(V)$ and $\operatorname{fin}(V)$. Let $\hat{\xi} = \xi + \mu(V)$ and $\hat{V} = \operatorname{fin}(V)/\mu(V)$, the quotient space. We can naturally define the usual norm $|\!|\!|\cdot|\!|\!|$ on $\hat{V}$ by $|\!|\!|\xi|\!|\!| = {}^{\circ}\|\xi\|$. A countably infinite sequence $\{\xi_i\}_{i\in\mathbf{N}}$ ($\xi_i \in \operatorname{fin}(V, \|\cdot\|)$) *approximately converges to* $\xi \in V$ in the norm $\|\cdot\|$ if

$$\forall \varepsilon \in \mathbf{R}^+\ \exists n \in \mathbf{N}\ \forall k \in \mathbf{N}, \quad k > n \ \Rightarrow\ \|\xi - \xi_i\| < \varepsilon. \quad (2)$$

A sequence $\{\xi_i\}_{i\in\mathbf{N}}$ ($\xi_i \in \operatorname{fin}(V, \|\cdot\|)$) is *S-*$\|\cdot\|$*-Cauchy* if

$$\forall \varepsilon \in \mathbf{R}^+\ \exists n \in \mathbf{N}\ \forall k, l \in \mathbf{N}, \quad k, l > n \ \Rightarrow\ \|\xi_k - \xi_l\| < \varepsilon. \quad (3)$$

A subset $X \subset \operatorname{fin}(V, \|\cdot\|)$ is *S-*$\|\cdot\|$*-complete* if for any S-$\|\cdot\|$-Cauchy sequence $\{\xi_i\}_{i\in\mathbf{N}}$, there exists $\xi \in X$ such that $\{\xi_i\}$ approximately converges to ξ in the norm $\|\cdot\|$.

The following result is a fundamental property of an internal normed space $(V, \|\cdot\|)$.

Theorem 1.1. *The subspace* $\operatorname{fin}(V)$ *is S-complete in* $\|\cdot\|$.

Corollary 1.2. (The Hull Completeness Theorem) $(\hat{V}, |\!|\!|\cdot|\!|\!|)$ *is a Banach space.*

Let $\mathcal{H}$ be an internal Hilbert space, and $T : \mathcal{H} \to \mathcal{H}$ an internal bounded linear operator such that the bound $\|T\|$ is finite. The bounded operator $\hat{T} : \hat{\mathcal{H}} \to \hat{\mathcal{H}}$, called the *standard part* of T, is defined by the relation $\hat{T}\hat{x} = \widehat{Tx}$ for any $x \in \operatorname{fin}(\mathcal{H})$.

For further information on nonstandard real analysis, we refer to Stroyan and Luxemburg [2] and Hurd and Loeb [4].

References

[1] C. C. Chang and H. J. Keisler. *Model Theory.* North-Holland, Amsterdam, 3rd edition edition, 1990.

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[3] A. Robinson and E. Zacon. A set-theoretical characterization of enlargements. In *Applications of Model Theory to Algebra, Analysis, and Probability*, pages 109–122, New York, 1969. Holt Reinehart and Winston.

[4] K. D. Stroyan and W. A. J. Luxemburg. *Introduction to the Theory of Infinitesimals.* Academic Press, New York, 1976.