

Informal and Intuitive introduction to nonstandard analysis

Hideyasu Yamashita

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1. Relativizing mathematical notions

Consider the following problem: Can the notion of *natural numbers* \mathbf{N} vary from person to person? More exactly, can the notion of natural numbers \mathbf{N}_A held by a person A be different from the other one \mathbf{N}_B held by another person B ? We suppose that \mathbf{N}_A and \mathbf{N}_B share the usual properties of natural numbers, especially the Peano axioms, including the mathematical induction principle. For example, the induction on \mathbf{N}_A is written as

$$(P(0) \ \& \ \forall n \in \mathbf{N}_A (P(n) \Rightarrow P(n+1))) \Rightarrow \forall n \in \mathbf{N}_A P(n) \quad (1)$$

for any property P .

At first glance, the answer seems to be “no”: we are lead to the following.

“Persuasive proof” of $\mathbf{N}_A = \mathbf{N}_B$: By the Peano axioms, we have $0 \in \mathbf{N}_A$ and $0 \in \mathbf{N}_B$. Let $n \in \mathbf{N}_A$. Clearly if $n \in \mathbf{N}_B$, then $n+1 \in \mathbf{N}_B$. By the induction, we have $\forall n \in \mathbf{N}_A n \in \mathbf{N}_B$. Thus $\mathbf{N}_A \subset \mathbf{N}_B$. Similarly we have $\mathbf{N}_A \supset \mathbf{N}_B$. Therefore $\mathbf{N}_A = \mathbf{N}_B$. *QED*

However, we find that there is much room for refutation of this “proof”. Notice the problematic phrase “for any property P ” in the induction principle.

We can rewrite the induction principle on \mathbf{N}_A as follows, without the phrase:

For any subset X of \mathbf{N}_A ,

$$(0 \in X \ \& \ \forall n \in \mathbf{N}_A \ (n \in X \Rightarrow n + 1 \in X)) \Rightarrow \forall n \in \mathbf{N}_A \ n \in X$$

or equivalently

For any subset X of \mathbf{N}_A ,

$$(0 \in X \ \& \ \forall n \in \mathbf{N}_A \ (n \in X \Rightarrow n + 1 \in X)) \Rightarrow X = \mathbf{N}_A.$$

Then one will find that the above “proof” implicitly assumed the absolute (i.e. not relative) notion of *subset*. If we relativize \mathbf{N} , we should relativize $\mathcal{P}(\mathbf{N})$ too, because, for a person’s perception, the concept of “all subsets of \mathbf{N} ” seems more abstract and more inaccessible than that of “all natural numbers”; one cannot have any *concrete* grasp of the uncountable set $\mathcal{P}(\mathbf{N})$. A concrete notion such as “the set of prime numbers less than 1000” seems an absolute (i.e. not relative) and objective notion, but it will be quite plausible that any conception of an infinite set, especially of an uncountable set, is subjective, and varies from person to person.

Let us relativize the induction principle. A ’s induction principle is

$$\text{For any } X \in \mathcal{P}_A(\mathbf{N}_A), \tag{2}$$

$$(0 \in X \ \& \ \forall n \in \mathbf{N}_A \ (n \in X \Rightarrow n + 1 \in X)) \Rightarrow X = \mathbf{N}_A.$$

B ’s induction principle is given similarly. We assume that A ’s zero $0_A \in \mathbf{N}_A$ is equal to B ’s zero $0_B \in \mathbf{N}_B$, i.e. $0_A = 0_B = 0$, and that both $\mathcal{P}_A(\mathbf{N}_A)$ and $\mathcal{P}_B(\mathbf{N}_B)$ are closed with respect to finite boolean operations:

$$X, Y \in \mathcal{P}_A(\mathbf{N}_A) \quad \Rightarrow \quad X \cap Y, X \cup Y, \mathbf{N}_A - X \in \mathcal{P}_A(\mathbf{N}_A),$$

$$X, Y \in \mathcal{P}_B(\mathbf{N}_B) \quad \Rightarrow \quad X \cap Y, X \cup Y, \mathbf{N}_B - X \in \mathcal{P}_B(\mathbf{N}_B).$$

The above argument of “The persuasive proof” for $\mathbf{N}_A = \mathbf{N}_B$ should be corrected as follows:

Proposition 1.1. The following conditions are equivalent.

- (i) $\mathbf{N}_A \in \mathcal{P}_B(\mathbf{N}_B)$.
- (ii) $\mathbf{N}_B \in \mathcal{P}_A(\mathbf{N}_A)$.
- (iii) $\mathcal{P}_A(\mathbf{N}_A) = \mathcal{P}_B(\mathbf{N}_B)$.
- (iv) $\mathbf{N}_B = \mathbf{N}_A$.

Proof. We will show (ii) \Rightarrow (iv) only. Suppose (ii), which implies $\mathbf{N}_B \subset \mathbf{N}_A$. Clearly $0 \in \mathbf{N}_B$, and

$$n \in \mathbf{N}_B \Rightarrow n + 1 \in \mathbf{N}_B (\subset \mathbf{N}_A). \quad (3)$$

Thus by A 's induction principle (2), it follows that $\mathbf{N}_A = \mathbf{N}_B$. *QED*

Actually, “The persuasive proof of $\mathbf{N}_A = \mathbf{N}_B$ ” is not the proof of $\mathbf{N}_A = \mathbf{N}_B$, but of (i) \Rightarrow (iv); the “proof” implicitly assumed that (i) is true. Thus it turned out that the relativization of the notion of “the set of all natural numbers” is logically possible: we may assume $\mathbf{N}_A \neq \mathbf{N}_B$ as well as $\mathbf{N}_A \notin \mathcal{P}_B(\mathbf{N}_B)$ and $\mathbf{N}_B \notin \mathcal{P}_A(\mathbf{N}_A)$. In the sequel we assume them.

We add an assumption that $\mathbf{N}_A \subset \mathbf{N}_B$. The consistence of the assumption is well-known in mathematical logic. For A , any number in $\mathbf{N}_B - \mathbf{N}_A$ is not a natural number, but an “hypernatural” one. It is called an *infinite hypernatural number* (for A). For any $n \in \mathbf{N}_B - \mathbf{N}_A$, B recognizes the interval $[0, n] \in \mathcal{P}_B(\mathbf{N}_B)$ as a finite set, but A recognizes it as an infinite set, because $\mathbf{N}_A \subset [0, n]$. Such sets are called *hyperfinite*.

The subsets of \mathbf{N}_B are classified into two types: internal or external. An *internal* set is an element of $\mathcal{P}_B(\mathbf{N}_B)$; all other subsets are called *external*. \mathbf{N}_A is an external subset of \mathbf{N}_B .

We can also relativize the notion of real numbers \mathbf{R} , and introduce \mathbf{R}_A and \mathbf{R}_B . In relativizing \mathbf{N} , we needed the relativization of the induction principle, which is accomplished by the relativization of $\mathcal{P}(\mathbf{N})$. As to \mathbf{R} , we need the relativization of the *completeness* in terms of Cauchy sequences, Dedekind cut, etc. It is also accomplished by relativizing $\mathcal{P}(\mathbf{R})$.

(to be continued)